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On the Approximation of a Continuum by Lemniscates

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For an arbitrary continuum E in the complex plane with connected complement Ω we study the rate of approximation of ∂E from outside by lemniscates in terms of level lines of a conformal mapping of Ω onto the exterior of the unit disk.

Key Words: lemniscates; Hilbert's theorem; quasiconformal curves; Faber polynomials.

1. INTRODUCTION

Let $E \subset \mathbb{C}$ be a continuum with connected complement $\Omega := \overline{\mathbb{C}} \setminus E$ with respect to the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. In 1897 D. Hilbert proved that if $L := \partial E$ is an analytic Jordan curve, then it can be approximated arbitrarily closely by lemniscates which lie in Ω and consist of one component only (for details, see [21]). Recently, Dolzhenko (cf. [9, p. 21]) raised the problem of estimating the rate of approximation of a closed Jordan curve by lemniscates in the Hausdorff metric in terms of properties of this curve.

In this paper we study the closeness of lemniscates to ∂E in terms of the behavior of level lines of the Riemann mapping Φ of Ω onto the exterior of the unit disk. Comparing our results with distortion theorems known in geometric function theory and the theory of quasiconformal mappings (see, for example, [18, 5, 14, 7, 16, 17, 4]) one can obtain further statements concerning Dolzhenko's problem. However, this purely geometrical (not approximational!) topic exceeds the scope of this paper.

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2. MAIN DEFINITIONS AND RESULTS

Let the function Φ map Ω conformally and univalently onto $\Delta := \{w : |w| > 1\}$, where Φ is normalized by the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Set $\Psi := \Phi^{-1}$ and let L_s denote the Jordan curve

$$L_s := \{z : |\Phi(z)| = 1 + s\} \qquad (s > 0).$$

Let p_n, $n \in \mathbb{N} := \{1, 2, ...\}$, be a polynomial of degree at most n. Denote by $J(p_n, c)$, $c > 0$, the lemniscate

$$J(p_n, c) := \{z : |p_n(z)| = c\}.$$

For a Jordan curve $\Gamma \subset \mathbb{C}$ denote by $\operatorname{int} \Gamma$ and $\operatorname{ext} \Gamma$ the bounded and unbounded components of $\overline{\mathbb{C}} \setminus \Gamma$. By $s_n(E)$ we denote the infimum of $s > 0$ for which there exists a polynomial $p_n = p_{n,s}$ such that $J(p_n, 1)$ is a Jordan curve satisfying the conditions

$$E \subset \operatorname{int} J(p_n, 1) \subset \operatorname{int} L_s. \tag{2.1}$$

By the Hilbert theorem (see [21, pp. 68–71]) applied to L_s

$$\lim_{n \to \infty} s_n(E) = 0.$$

The rate of the decrease of $s_n(E)$ as $n \to \infty$ is the main topic of this paper.

THEOREM 1. *Let $E \subset \mathbb{C}$ be an arbitrary continuum with connected complement. Then the inequality*

$$s_n(E) \leqslant c_1 \frac{\log n}{n} \qquad (n > 1) \tag{2.2}$$

holds with some $c_1 > 0$ independent of n.

If we know more about the geometry of E, then the estimate (2.2) can be sharpened.

Let Γ be a rectifiable Jordan curve $z = z(s)$ with arc length $s \in [0, |\Gamma|]$, where $|\Gamma|$ denotes the length of Γ. If $\arg z'(s)$ can be defined on $[0, |\Gamma|]$ to become a function of bounded variation, then Γ is said to be of *bounded rotation* (see [18, p. 63]).

For $z = z(s) \in \Gamma$, we consider the function

$$h(\zeta) := \arg(\zeta - z) \qquad (\zeta \in \Gamma),$$

where ζ starts at $z(s+)$ and stops at $z(s-)$. If there is a fixed constant $c_2>0$ such that the total variation $\mathrm{Var}_\zeta\, h(\zeta)$ of $h(\zeta)$ as a function of ζ satisfies

$$\mathrm{Var}_\zeta \arg(\zeta - z) \leqslant c_2 \qquad (z \in \Gamma),$$

then Γ is said to be of *bounded secant variation* (see [2, 13]). Curves of bounded rotation and curves consisting of a finite number of Dini-smooth arcs are typical particular cases of curves of bounded secant variation (for details, see [2, 13, 11]).

THEOREM 2. *Let ∂E be a Jordan curve of bounded secant variation. Then the inequality*

$$s_n(E) \leqslant \frac{c_3}{n} \qquad (n \in \mathbb{N}) \tag{2.3}$$

holds with some $c_3>0$ independent of n.

Proofs of Theorems 1 and Theorem 2, given in Section 4, are based essentially on the estimation of the maximum, taken over Ω, of the difference between the nth degree of the Riemann mapping function $\Phi(z)$ and the nth Faber polynomial.

Note that in Theorems 1 and Theorem 2 n is given and one looks for p_n and s such that (2.1) is true. The statement: *Given $s>0$, one looks for n such that there exists p_n for which* (2.1) *is true*, would be similar. Obviously, the minimal n with this property, denoted by $n(s)$, satisfies

$$n(s) \leqslant \frac{c_4}{s} \log \frac{1}{s} \qquad \left(0 < s \leqslant \frac{1}{2}\right)$$

for Theorem 1 and

$$n(s) \leqslant \frac{c_5}{s} \qquad \left(0 < s \leqslant \frac{1}{2}\right)$$

for Theorem 2, where c_j, $j=4, 5$, are independent of n.

Next we are going to discuss the sharpness of (2.3). Analysis of the proof of Theorem 2 shows that for some E (for example for a disk or, more generally, for a domain bounded by an analytic curve) the quantity $s_n(E)$ is much smaller than $O(1/n)$. However, if L has at least at one point something like an angle $<\pi$ with respect to E, then the rate of decrease of $s_n(E)$ found in Theorem 2 cannot be improved. Below we explain this effect more precisely.

We restrict our consideration to the case when L is a quasiconformal Jordan curve [1, 15]. Ahlfors (cf. [15, p. 104]) has established a geometrical criterion for quasiconformality of a curve which can be formulated as follows: A Jordan curve Γ is quasiconformal if and only if for any pair of points z_1 and $z_2 \in \Gamma$ the inequality

$$\min\{\operatorname{diam} \Gamma', \operatorname{diam} \Gamma''\} \leqslant c\,|z_1 - z_2|$$

holds with some constant $c = c(L) \geqslant 1$, where Γ' and Γ'' are the two arcs which $\Gamma \setminus \{z_1, z_2\}$ consists of.

Using Ahlfors' criterion one can easily verify that curves of bounded rotation without cusps and piecewise smooth curves without cusps are quasiconformal.

Suppose L is quasiconformal and let $z \in L$, $r > 0$; we denote by $\gamma_z(r) \subset \Omega$ an arc of the circle $\{\zeta : |\zeta - z| = r\}$ that separates z from ∞ in Ω (i.e., $\gamma_z(r)$ has nonempty intersection with every Jordan arc in Ω that joins z to ∞). If $\gamma_z(r)$ is not uniquely determined, we agree to choose it so that, in the division of Ω into two subdomains by $\gamma_z(r)$, the unbounded domain is as large as possible for given z and r.

If $0 < r < R < (\operatorname{diam} L)/2$, then $\gamma_z(r)$ and $\gamma_z(R)$ are opposite sides of a quadrilateral $Q_z(r, R) \subset \Omega$ whose other two sides are the parts of L which connect the ends of $\gamma_z(r)$ and $\gamma_z(R)$. We denote by $m_z(r, R)$ the module of this quadrilateral, i.e., the module of the family of arcs that separate the sides $\gamma_z(r)$ and $\gamma_z(R)$ in $Q_z(r, R)$ (see [1, 15]).

THEOREM 3. *Let $L = \partial E$ be a quasiconformal curve and suppose there exists a point $\zeta \in L$ such that*

$$\lim_{\substack{t \to 0+ \\ r \to 0+}} \left(\frac{1}{\pi} \log \frac{1}{t} - m_\zeta(tr, r) \right) = \infty. \tag{2.4}$$

Then the inequality

$$s_n(E) \geqslant \frac{c_6}{n} \qquad (n \in \mathbb{N}) \tag{2.5}$$

holds with some constant $c_6 > 0$ independent of n.

As an example for (2.4) assume that there exists a circular sector with center at ζ, radius $\delta > 0$ and opening $\beta\pi$, $1 < \beta < 2$ in $\bar{\Omega}$. Then by the comparison principle for a module we have for $0 < r < R < \delta$,

$$m_\zeta(r, R) \leqslant \frac{1}{\beta\pi} \log \frac{R}{r},$$

and therefore (2.4) is fulfilled.

It is worth pointing out that under the hypotheses of Theorem 3, if, moreover, L is of bounded rotation, then by [12, inequalities (1) and (3)]

$$\int_{tr}^{r} \frac{dx}{|\gamma_\zeta(x)|} \leqslant m_\zeta(tr, r) \leqslant \int_{tr}^{r} \frac{dx}{|\gamma_\zeta(x)|} + c_7,$$

where $c_7 = c_7(L) > 0$. Hence, (2.4) is in this case equivalent to the condition

$$\lim_{\substack{t \to 0+ \\ r \to 0+}} \left(\frac{1}{\pi} \log \frac{1}{t} - \int_{tr}^{r} \frac{dx}{|\gamma_\zeta(x)|} \right) = \infty, \tag{2.6}$$

which is easier to verify.

Note that if L is smooth, then (2.6) is not fulfilled in general. So, the problem of improving (2.3) in this case remains open.

In what follows we denote by $c, c_1, \ldots$ positive constants (different each time, in general) that either are absolute or depend on parameters not essential for the arguments; sometimes such a dependence will be indicated.

For positive a and b we use the expression $a \preccurlyeq b$ (order inequality) if $a \leqslant cb$. The expression $a \asymp b$ means that $a \preccurlyeq b$ and $b \preccurlyeq a$ simultaneously.

Set for $z \in \mathbb{C}$ and $A \subset \mathbb{C}$,

$$d(z, A) := \operatorname{dist}(z, A) := \inf_{\zeta \in A} |z - \zeta|.$$

3. AUXILIARY RESULTS FROM GEOMETRIC FUNCTION THEORY AND THE THEORY OF QUASICONFORMAL MAPPINGS

Let $E \subset \mathbb{C}$ be an arbitrary continuum (with connected complement). We begin with a general distortion property of the mapping Φ, which follows easily from the Koebe one-quarter-theorem. Namely, for $z \in \Omega \setminus \{\infty\}$ we have

$$\frac{1}{4} \frac{|\Phi(z)| - 1}{d(z, L)} \leqslant |\Phi'(z)| \leqslant 4 \frac{|\Phi(z)| - 1}{d(z, L)} \tag{3.1}$$

(see, for example, [5, p. 58]). Therefore, if $z, \zeta \in \Omega \setminus \{\infty\}$, $w := \Phi(z)$, $t := \Phi(\zeta)$ satisfy $|w - t| \leqslant (|w| - 1)/2$, then

$$|\zeta - z| \geqslant \frac{1}{16} \frac{d(z, L)}{|w| - 1} |w - t| \tag{3.2}$$

(cf. [6, Lemma 1]).

We will need the following direct consequence of (3.2): If $|w|-1 \geqslant 2s$, then

$$d(z, L_s) \geqslant \tfrac{1}{32}\, d(z, L). \tag{3.3}$$

Next we recall in a form convenient for us a result which is due to Tamrazov [20] (see also [5, p. 186]).

LEMMA 1. *Let the function f be analytic in a domain $G \subset \mathbb{C}$ and continuous on $\bar{G} \subset \mathbb{C}$. If for some $M>0$, $z_0 \in \partial G$, $m \in \mathbb{N}$ and $\rho > 0$ the inequality*

$$|f(z)| \leqslant M\left(1+\left|\frac{z-z_0}{\rho}\right|^m\right) \qquad (z \in \partial G) \tag{3.4}$$

holds, then

$$|f(z)| \leqslant c\, M \qquad (z \in G,\ |z-z_0| \leqslant \rho),$$

where $c = c(m)$.

In the rest of this section we assume that $E = \bar{G}$ is bounded by a quasiconformal curve $L := \partial G$. In this case the conformal mapping Φ can be extended to a quasiconformal mapping $\Phi\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ (see [1, Chap. 3]). Therefore the following result can be derived from the distortion properties of quasiconformal homeomorphisms of the plane (for details, see [3, Lemma 1] or [5, pp. 97–98]).

LEMMA 2. *Let $\zeta_j \in \bar{\Omega}\setminus\{\infty\}$, $w_j := \Phi(\zeta_j)$, $j = 1, 2, 3$. Then:*

(i) *the conditions $|\zeta_1-\zeta_2| \leqslant c_1\, |\zeta_1-\zeta_3|$ and $|w_1-w_2| \leqslant c_2\, |w_1-w_3|$ are equivalent; besides, the constants c_1 and c_2 are mutually dependent and depend on L;*

(ii) *if $|\zeta_1-\zeta_2| \leqslant c_1\, |\zeta_1-\zeta_3|$, then*

$$c_3 \left|\frac{w_1-w_3}{w_1-w_2}\right|^{\alpha} \leqslant \left|\frac{\zeta_1-\zeta_3}{\zeta_1-\zeta_2}\right| \leqslant c_4 \left|\frac{w_1-w_3}{w_1-w_2}\right|^{1/\alpha}, \tag{3.5}$$

where $0 < \alpha = \alpha(L) \leqslant 1$, $c_j = c_j(c_1, L) > 0$, $j = 3, 4$.

For $z \in \Omega\setminus\{\infty\}$ set $z_L := \Psi(\Phi(z)/|\Phi(z)|)$ and denote by $z^* \in L$ one of the points with the property

$$|z - z^*| = d(z, L).$$

Application of the first part of Lemma 2 to the triplet of points z, z^*, and z_L implies

$$d(z, L) \asymp |z - z_L|. \tag{3.6}$$

Further, we claim that there exist constants $c_5 = c_5(L)$ and $m = m(L) \in \mathbb{N}$ such that for any $z, \zeta \in L_s$, $0 < s < 1$,

$$\frac{1}{d(\zeta, L)} \leqslant \frac{c_5}{d(z, L)} \left(1 + \left|\frac{\zeta - z}{d(z, L)}\right|^m\right). \tag{3.7}$$

Indeed, if $|\Phi(\zeta) - \Phi(z)| \leqslant s$, then by the first part of Lemma 2 and (3.6) we have

$$d(\zeta, L) \asymp |\zeta_L - \zeta| \asymp |\zeta_L - z| \asymp |z_L - z| \asymp d(z, L),$$

which implies (3.7) with an arbitrary m.

Let $|\Phi(\zeta) - \Phi(z)| > s$. By virtue of (3.5) we obtain

$$\begin{aligned} \left|\frac{\zeta - z}{\zeta - \zeta_L}\right| &\preccurlyeq \left|\frac{\Phi(\zeta) - \Phi(z)}{\Phi(\zeta) - \Phi(\zeta_L)}\right|^{1/\alpha} \\ &= \left|\frac{\Phi(z) - \Phi(\zeta)}{\Phi(z) - \Phi(z_L)}\right|^{1/\alpha} \preccurlyeq \left|\frac{z - \zeta}{z - z_L}\right|^{1/\alpha^2}. \end{aligned}$$

Therefore, in this case (3.7) follows from (3.6) with any $m \geqslant 1/\alpha^2 - 1$.

Next, we cite a result that is essentially due to Belyi and Mikljukov [8].

Lemma 3. *Let* $z_0 \in L$, $z_1, z_2 \in \Omega$, $0 < |z_1 - z_0| < |z_2 - z_0| < (\operatorname{diam} L)/2$. *Then with the notation of Section* 2

$$\exp\{\pi m_{z_0}(|z_0 - z_1|, |z_0 - z_2|)\} \geqslant c_1 \left|\frac{\Phi(z_2) - \Phi(z_0)}{\Phi(z_1) - \Phi(z_0)}\right|, \tag{3.8}$$

where $c_1 = c_1(L)$.

Proof. Let $w_j := \Phi(z_j)$, $j = 0, 1, 2$; $r := |z_0 - z_1|$, $R := |z_0 - z_2|$. Denote by $\Gamma = \Gamma(z_0, z_1, z_2, \Omega)$ the family of all crosscuts of Ω (i.e., Jordan arcs in Ω with ends on L), which separate z_0 and z_1 from z_2 and ∞ in Ω. Since the module of a family of curves is a conformal invariant (cf. [1, 15]) Lemma 2 from [7] (see also [5, Lemma 2.2, p. 36]) and Lemma 2 imply

$$\frac{1}{\pi} \log\left|\frac{w_0 - w_2}{w_0 - w_1}\right| - c_2 \leqslant m(\Gamma).$$

Further, an immediate consequence of the first part of Lemma 2 is the inequality

$$m(\Gamma) \leqslant m_{z_0}(r, R) + c_3 .$$

Comparing the last two inequalities we get (3.8). ∎

4. FABER POLYNOMIALS. PROOFS OF THEOREMS 1 AND 2

First we review some of the standard facts about Faber polynomials; see, for example, [19, 10]. The *Faber polynomials* $F_n(z)$ are defined by a generating function

$$\frac{w\,\Psi'(w)}{\Psi(w)-z} = 1 + \sum_{n=1}^{\infty} \frac{F_n(z)}{w^n} \qquad (|w| > 1,\ z \in E).$$

For an arbitrary continuum E with connected complement and $n > 1$,

$$\|F_n\|_E := \max_{z \in E} |F_n(z)| \preccurlyeq (n \log n)^{1/2} \tag{4.1}$$

(cf. [19, p. 136]). At the same time there exists a quasiconformal curve $L = \partial E$ such that for some $\alpha > 0$ and a subsequence $\Lambda \subset \mathbb{N}$,

$$\|F_n\|_E \succcurlyeq n^{\alpha} \qquad (n \in \Lambda) \tag{4.2}$$

(see [11]).

Further, if $L = \partial E$ is of bounded secant variation, then

$$\|F_n\|_E \preccurlyeq 1 \qquad (n \in \mathbb{N}) \tag{4.3}$$

(see [2, 13]).

Recall that $F_n(z)$ is the polynomial part of $\Phi(z)^n$. In the reasoning below the function

$$\omega_n(z) := \Phi(z)^n - F_n(z) \qquad (z \in \Omega),$$

which is analytic in Ω and satisfies $\omega_n(\infty) = 0$, plays a key role.

Proof of Theorem 1. By the maximum principle and (4.1),

$$|\omega_n(z)| \preccurlyeq (n \log n)^{1/2} \qquad (z \in \Omega).$$

Therefore, there exists $c>0$ such that for $z\in\Omega$ satisfying $|\Phi(z)|\geqslant 1+(c\log n)/n$ we have

$$\tfrac{1}{2}\,|\Phi(z)|^n\leqslant|F_n(z)|\leqslant 2\,|\Phi(z)|^n. \tag{4.4}$$

Consider the lemniscate

$$J_n:=J\left(F_n,2\left(1+c\,\frac{\log n}{n}\right)^n\right).$$

According to (4.4) all zeros of F_n belong to int $L_{(c\log n)/n}$ and

$$L_{(c\log n)/n}\subset\overline{\operatorname{int} J_n}\,. \tag{4.5}$$

Therefore, the level set J_n of F_n consists of a finite number of disjoint analytic curves. Since by the minimum principle, each component of J_n must contain a zero of F_n, it follows that J_n is a single Jordan curve.

At the same time for $z\in\Omega$ satisfying $|\Phi(z)|\geqslant 1+(2c\log n)/n$ and $n>n_0$ large enough we obtain according to (4.4),

$$\frac{|F_n(z)|}{\left(1+c\,\dfrac{\log n}{n}\right)^n}\geqslant\frac{1}{2}\left(1+\frac{c\,\dfrac{\log n}{n}}{1+c\,\dfrac{\log n}{n}}\right)^n>2.$$

Therefore, for $n>n_0$,

$$J_n\subset\operatorname{int} L_{(2c\log n)/n}\,. \tag{4.6}$$

Comparing (4.5) and (4.6) we get (2.2). ∎

In the proof of Theorem 1 we approximate ∂E by lemniscates given by Faber polynomials. The following example shows that the rate of such an approximation as found in (4.6) is best possible even in the class of all domains bounded by a quasiconformal curve.

Example. Let L be the quasiconformal curve constructed by Gaier [11], which is satisfying (4.2). Suppose that $s=s_n>0$ and $c_n>0$ are such that

$$L\subset\operatorname{int} J_n\subset\operatorname{int} L_s,$$

where

$$J_n:=J(F_n,c_n).$$

According to (4.2) and using the maximum principle, $c_n \geqslant n^\alpha$ for $n \in \Lambda$. Note that the function $F_n(z)/\Phi(z)^n$ is analytic in Ω and is equal to 1 at the interior point ∞. Therefore, by the minimum principle applied on the exterior of J_n, there exists $z_n \in J_n$ such that

$$|\Phi(z_n)|^n \geqslant |F_n(z_n)| = c_n \geqslant n^\alpha;$$

that is,

$$|\Phi(z_n)| \geqslant \exp\left(\alpha \frac{\log n}{n}\right) \geqslant 1 + \alpha \frac{\log n}{n}.$$

This means that

$$s \geqslant \alpha \frac{\log n}{n},$$

which shows the sharpness of the factor $(\log n)/n$ in (4.6).

Proof of Theorem 2. By (4.3) and the maximum principle,

$$|\omega_n(z)| \leqslant 1 \qquad (z \in \Omega).$$

The rest of the proof runs as before, i.e., by modifying the reasoning from the proof of Theorem 1 in an obvious way (taking $1/n$ instead of $(\log n)/n$). ▮

5. PROOF OF THEOREM 3

Suppose s satisfies (2.1). There is no loss of generality in assuming that $s < 1/(2n)$ and that the degree of p_n is equal to n, where p_n is a polynomial as in (2.1). Let $\phi_n(z)$ be a holomorphic branch of $p_n(z)^{1/n}$ which gives a conformal map of ext $J(p_n, 1)$ onto Δ. Set $\psi_n := \phi_n^{-1}$.

Applying the Schwarz lemma to the functions

$$\frac{1}{\Phi\left(\psi_n\left(\frac{1}{t}\right)\right)} \qquad (|t| < 1)$$

and

$$\frac{1}{\phi_n\left(\Psi\left(\frac{1+s}{z}\right)\right)} \qquad (|z| < 1),$$

we obtain for $|u|>1+s$, after we invert the map $w=\Phi(\psi_n(1/t))$, that

$$\frac{|u|}{1+s}\leqslant|\phi_n(\Psi(u))|\leqslant|u|. \tag{5.1}$$

Consider an arbitrary point $u=u(n)$ with $|u|=1+2/n$ and set $z:=\Psi(u)$. We will omit the "n" in our notations when no confusion may arise. By (3.1), (3.3), and (5.1),

$$\frac{1}{n}\,|p_n(z)|^{1/n-1}\,|p_n'(z)|=|\phi_n'(z)|\leqslant\frac{8}{n\,d(z,L_s)}\preccurlyeq\frac{1}{n\,d(z,L)}.$$

Thus for any $\zeta\in L_{2/n}$, by the second part of (5.1) for $\Psi(u)=\zeta$ instead of z and by (3.7), the inequality

$$|p_n'(\zeta)|\preccurlyeq\frac{1}{d(\zeta,L)}\preccurlyeq\frac{1}{d(z,L)}\left(1+\left|\frac{\zeta-z}{d(z,L)}\right|^m\right) \tag{5.2}$$

holds with some $m=m(L)\in\mathbb{N}$.

Let v be such that

$$\frac{u}{v}>0,\qquad |v|=1+2s. \tag{5.3}$$

According to Lemma 2, (3.6), and (5.3) for $\xi:=\Psi(v)$ and $z_L:=\Psi(u/|u|)$ we have

$$|\xi-z|\preccurlyeq|z_L-z|\preccurlyeq d(z,L).$$

Therefore, Lemma 1 and (5.2) yield that

$$|p_n'(\xi)|\preccurlyeq\frac{1}{d(z,L)},$$

which by virtue of (3.1), (5.1), and (5.3) implies that

$$\begin{aligned}\frac{s}{4(1+s)\,d(\xi,L)}&=\frac{1}{4}\left(\frac{2s}{1+s}-1\right)\frac{1}{d(\xi,L)}\leqslant|\phi_n'(\xi)|\\&=\frac{1}{n}\,|p_n'(\xi)|\,|p_n(\xi)|^{1/n-1}\leqslant\frac{1}{n}\,|p_n'(\xi)|\preccurlyeq\frac{1}{n\,d(z,L)}.\end{aligned}$$

Further, taking into account (3.6), we have

$$\left|\frac{z-z_L}{\xi-z_L}\right|\preccurlyeq\frac{1}{s\,n}. \tag{5.4}$$

Next, we choose z such that $z_L = \zeta$, where ζ is the point from (2.4). We can assume that $|\xi - \zeta| < |z - \zeta|$ because otherwise by Lemma 2, $s \geqslant 1/n$ which corresponds to (2.5). The inequality (5.4) and Lemma 3 imply that

$$\left|\frac{z-\zeta}{\xi-\zeta}\right| \leqslant \left|\frac{\Phi(z)-\Phi(\zeta)}{\Phi(\xi)-\Phi(\zeta)}\right| \leqslant \exp(\pi\, m_\zeta(|\xi-\zeta|, |z-\zeta|)). \tag{5.5}$$

Now let us assume contrary to (2.5) that

$$\liminf_{n\to\infty} n\, s_n(E) = 0; \tag{5.6}$$

i.e., there exists a sequence $\Lambda \subset \mathbb{N}$ such that

$$n\, s_n(E) \to 0 \qquad \text{as} \quad \Lambda \ni n \to \infty.$$

By the second part of Lemma 2 for $z = z(n, \zeta)$, $\xi = \xi(s, \zeta)$, $s := 2s_n(E)$ constructed above,

$$t_n := \left|\frac{\xi-\zeta}{z-\zeta}\right| \to 0 \qquad \text{as} \quad \Lambda \ni n \to \infty,$$

$$r_n := |z-\zeta| \to 0 \qquad \text{as} \quad \Lambda \ni n \to \infty,$$

and by (5.5),

$$\frac{1}{\pi} \log \frac{1}{t_n} - m_\zeta(t_n r_n, r_n) \leqslant c,$$

which contradicts our assumption (2.4). Hence, (5.6) is false and we obtain (2.5). ∎

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